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COMMENT

Comment on 'Almost-periodic time observables for bound quantum systems'

Eric A Galapon

Theoretical Physics Group, National Institute of Physics, University of the Philippines, Diliman Quezon City 1101, Philippines

E-mail: eric.galapon@up.edu.ph

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Abstract

Hall (2008 *J. Phys. A: Math. Theor.* **41** 255301) makes two claims on a time operator constructed in Galapon (2002 *Proc. R. Soc.* A **458** 2671). We discuss why both claims are wrong.

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In [1], a time operator $\hat{T}$ is constructed solely from the eigenvectors and eigenvalues of a given Hamiltonian $\hat{H}$. The Hamiltonian is assumed to have a discrete non-degenerate spectrum E_k with the corresponding eigenvectors $|E_k\rangle, k = 1, 2, 3, \ldots$; moreover, the Hamiltonian is semibounded from below such that $0 < E_1 < E_2 < E_3 < \ldots$, and the eigenvalues satisfy $\sum_{k=1}^{\infty} E_k^{-2} < \infty$. The $|E_k\rangle$'s form a complete set and span the entire Hilbert space $\mathcal{H}$, which allows us to write every vector in $\mathcal{H}$ in the form $|\psi\rangle = \sum_{k=1}^{\infty}\langle E_k|\psi\rangle|E_k\rangle$. The domain $\mathcal{D}_H$ of the Hamiltonian $\hat{H}$ consists of vectors $|\phi\rangle = \sum_{k=1}^{\infty} a_k|E_k\rangle$ such that $\sum_{k=1}^{\infty} E_k^2|a_k|^2 < \infty$. The time operator $\hat{T}$ is given by

$$\hat{T} = \mathrm{i}\hbar \sum_{j\neq k} \frac{|E_j\rangle\langle E_k|}{E_j - E_k}. \tag{1}$$

The domain $\mathcal{D}_T$ of $\hat{T}$ consists of vectors of the form $|\varphi\rangle = \sum_{k=1}^{N} b_k|E_k\rangle$ for some finite but otherwise arbitrary positive integer N. The Hamiltonian eigenvectors $|E_k\rangle$ belong to $\mathcal{D}_T$; this implies that $\mathcal{D}_T$ is dense or $\hat{T}$ is densely defined. The operators $\hat{H}$ and $\hat{T}$ satisfy the canonical commutation relation $[\hat{T}, \hat{H}]|\varphi\rangle = \mathrm{i}\hbar|\varphi\rangle$ for all $|\varphi\rangle$ in $\mathcal{D}_T$ such that $\sum_k b_k = 0$; these vectors comprise the subspace which we referred to as the canonical domain $\mathcal{D}_c$ of $\hat{H}$ and $\hat{T}$.

Now Hall claims that $[\hat{T}, \hat{H}]|E_k\rangle = 0$ for every k, and that $\mathcal{D}_c$ is not dense. These claims are flagrantly erroneous.

To assert that $[\hat{T}, \hat{H}]|E_k\rangle = 0$ one implies that $|E_k\rangle$ belongs to the commutator domain of $\hat{H}$ and $\hat{T}$, in particular, $|E_k\rangle$ belongs to the domain of $\hat{H}\hat{T}$. But $|E_k\rangle$ does not. Since $|E_k\rangle$ belongs to $\mathcal{D}_T$ for all $|E_k\rangle$'s, we have

$$\hat{T}|E_k\rangle = \sum_{j\neq k} \frac{\mathrm{i}\hbar}{(E_j - E_k)}|E_j\rangle, \tag{2}$$

and in order for $\hat{T}|E_k\rangle$ to belong to the domain of $\hat{H}$, it must be that $\sum_{j\neq k} E_j^2/(E_j - E_k)^2 < \infty$, which cannot be satisfied because $E_j^2/(E_j - E_k)^2 \to 1$ as $j \to \infty$ for a fixed k. Hence the expression $[\hat{T}, \hat{H}]|E_k\rangle$ does not make sense in the Hilbert space. This prevents us from arriving at the contradiction $\langle E_k|[\hat{T}, \hat{H}]|E_k\rangle = 0 = i\hbar\langle E_k|E_k\rangle$, which what Hall may be trying to say because we cannot have the equality $[\hat{T}, \hat{H}]|E_k\rangle = 0$, even formally.

Also to assert that $\mathcal{D}_c$ is not dense one implies that there exists a vector $|\psi\rangle \neq \mathbf{0}$ in the Hilbert space $\mathcal{H}$ that is orthogonal to all vectors in $\mathcal{D}_c$, that is, $\langle\psi|\varphi\rangle = 0$ for all $|\varphi\rangle$ in $\mathcal{D}_c$; otherwise, $\mathcal{D}_c$ is dense. To show that $\mathcal{D}_c$ is dense, it is sufficient to demonstrate that $\mathcal{D}_c$ itself has a dense subspace. Let $\mathcal{D}_c'$ be the linear span of the vectors $|\psi_{m,n}\rangle = |E_m\rangle - |E_n\rangle$ for all positive integers m, n; $\mathcal{D}_c'$ is clearly a subspace of $\mathcal{D}_c$. Now let $|\psi\rangle \neq 0$ be in $\mathcal{H}$ and orthogonal to all $|\varphi\rangle$ in $\mathcal{D}_c'$. Then it must be that $\langle\psi_{m,n}|\psi\rangle = 0$, which implies the equality $\langle E_m|\psi\rangle = \langle E_n|\psi\rangle$ for all m, n. Since $|\psi\rangle \neq 0$ there is at least an $n = n_0$ such that $\langle E_{n_0}|\psi\rangle \neq 0$. Then $\langle E_k|\psi\rangle = \langle E_{n_0}|\psi\rangle$ for all $k = 1, 2, \ldots$; or $|\psi\rangle = \sum_{k=1}^{\infty}\langle E_{n_0}|\psi\rangle|E_k\rangle = \langle E_{n_0}|\psi\rangle\sum_{k=1}^{\infty}|E_k\rangle$, which does not belong to $\mathcal{H}$. Then the only way for $|\psi\rangle$ to be simultaneously in $\mathcal{H}$ and orthogonal to $\mathcal{D}_c$ is for $|\psi\rangle = 0$. Hence $\mathcal{D}_c$ is dense. Hall's 'proof' of the non-denseness of $\mathcal{D}_c$ is a misunderstanding of the definition of a dense subspace.

References

[1] Galapon E A 2002 *Proc. R. Soc.* A **487** 2671

[2] Hall M J W 2008 *J. Phys. A: Math. Theor.* **41** 255301